Signal Analysis & Communication ECE355

Ch. 3.2: Response of LTI Systems to Complex Exponentials.

Lecture 13

05-10-2023

Ch. 3.2: RESPONSE OF LTI TO COMPLEX EXPONENTIALS

Recall:

(I) - The representation & analysis of LTI systems through the convolution sum (DT) is based on representing signals as linear combination of shifted impulses.

$$x_{[k]} = \sum_{n=-\infty}^{\infty} x_{[n]}\, \delta_{[n-k]}$$

Sifting property of impulse.

OR

$$x_{[n]} = \sum_{k=-\infty}^{\infty} x_{[k]}\, \delta_{[n-k]}$$

- Here we represent signals as linear combination of complex exponentials.

Fourier Series & Fourier Transform.
(DT) (CT).

- These can be used to construct broad & useful classes of signals.

(II) - Because of the linearity property (superposition), the response of the LTI sys. to any I/P consisting of a linear combination of basic signals is the same linear combination of the individual responses of the basic signals.

- These responses were all shifted versions of unit impulse responses

- Here, the response of an LTI sys. to complex exponential also, has a "particularly simple form", which provides us with another "convenient representation" of LTI systems.

To analyze LTI sys. & gain insights via their properties!

- We are starting with the CT case first.

$$x_{(t)} = e^{st} \longrightarrow \boxed{h_{(t)}} \longrightarrow y_{(t)} = ?$$

$s \in \mathbb{C}$

- Let's first see the CT LTI sys.'s response with complex exp. I/p.

$$y_{(t)} = \int_{-\infty}^{\infty} h_{(\tau)} e^{s(t-\tau)} d\tau$$

Conv. Commutative Property.

$$= e^{st} \boxed{\int_{-\infty}^{\infty} h_{(\tau)} e^{-s\tau} d\tau}$$

- Define

$$\boxed{H_{(s)} \triangleq \int_{-\infty}^{\infty} h_{(\tau)} e^{-s\tau} d\tau}$$

" Laplace Transform of $h_{(\cdot)}$ "

" Transfer Function of LTI Sys."

THEOREM: If $x_{(t)} = e^{st}$ & $H_{(s)}$ exists

then $\boxed{y_{(t)} = H_{(s)} e^{st}}$

- Two interpretations:

① Same complex exponential form as of I/p $\underbrace{e^{st}}_{\text{called Eigen function of LTI sys.}}$

② Scaled by a constant $\underbrace{H_{(s)}}_{\text{Eigen value.}}$

COROLLARY: If the I/p to an LTI sys. with impulse response $h_{(t)}$ is

$$x_{(t)} = \sum_k a_k e^{s_k t}$$

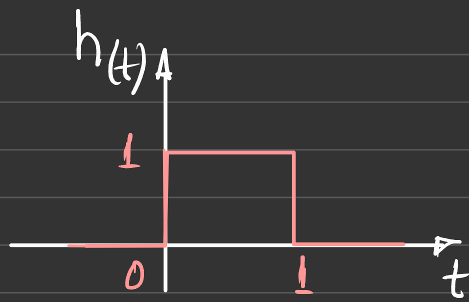
Linear combination of complex exp. functions

then

$$\boxed{y_{(t)} = \sum_k a_k H_{(s_k)} e^{s_k t}} \text{ — (A)}$$

Example:

$$h(t) = u(t) - u(t-1)$$
$$x(t) = \cos(4t)$$



- To find $y(t)$ via eqn. (A):
  - I - Find $H(s)$
  - II - Express $x(t)$ as linear combination of complex exp.

- I. $$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$
$$= \int_{0}^{1} e^{-s\tau} d\tau = \begin{cases} \frac{1}{s}(1-e^{-s}) & , s \neq 0 \\ 1 & , s = 0 \end{cases}$$

- II. $$x(t) = \frac{1}{2}\left[e^{j4t} + e^{-j4t}\right]$$

- Using eqn (A)
$$y(t) = \frac{1}{2}\left[\frac{1}{j4}(1-e^{-j4})e^{j4t} + \frac{1}{-j4}(1-e^{j4})e^{-j4t}\right]$$
$$= \frac{1}{4}\left[\sin(4t) - \sin(4t-4)\right]$$

- **DT Case:**

- Assume $$e^{sn} = z^n \quad \text{where} \quad z \triangleq e^{s} \; ; \; z \in \mathbb{C}$$

$$z^n \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

- LTI

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, z^{n-k}$$

Comm. property of conv.

$$= z^n \sum_{k=-\infty}^{\infty} h[k]\, z^{-k}$$

- Define

$$\boxed{H(z) = \sum_{k=-\infty}^{\infty} h[k]\, z^{-k}}$$

- If $H(z)$ exists, then

$$y[n] = \underbrace{H(z)}_{\text{Eigen value}}\ \underbrace{z^n}_{\text{Eigen function.}}$$

NOTE 1: $(e^x)^y \neq e^{xy}$ for general $x, y \in \mathbb{C}$

But $(e^x)^n = e^{xn}$ for $n \in \mathbb{Z}$

NOTE 2: In Fourier Series/Transforms, we focus on sinusoidal complex exponentials of the form:

CT: $e^{j\omega t}$; DT: $e^{j\omega n}$

CT: Periodic with fundamental period $\frac{2\pi}{|\omega|}$

DT: Periodic if $\omega = \frac{2\pi k}{N}$ for some $k, N \in \mathbb{Z}$

- CONCLUSION:

- For both CT & DT, if the I/P to an LTI sys. is represented as a linear combination of complex exponentials, then the O/P can also be represented as a linear combination of THE SAME complex exponential signals.